The Pareto principle and resource egalitarianism

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Highlights

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- An alternative partial characterization of the consensual Rawlsian social ordering (Sprumont (2012)) is provided.

- The leximin Paretian ordering is introduced and characterized.

- This paper contributes to the line of research initiated by Sprumont (2012) through the introduction of the leximin Paretian ordering satisfying a stronger bundle-reducing principle and a stronger Paretian axiom that are mutually compatible.
The Pareto principle and resource egalitarianism

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Abstract

This paper introduces and studies the \textit{leximin Paretian ordering}, which refines the consensual leximin ordering by adding the Pareto principle to the concept of lexicographic egalitarianism. We also provide an alternative characterization of the consensual Rawlsian ordering. We introduce several new axioms, including the Permutation Pareto Principle and Internal Dominance, and study their logical relationships.

\textit{Keywords:} Pareto principle, social ordering, egalitarianism

1. Introduction

In this paper, we define and study egalitarianism in the context of social ordering and in a multi-commodity model with a fixed number of agents. Social ordering addresses the question of how we specify the aim of the society which aggregates individual preferences to construct an ordering over all conceivable allocations. Egalitarianism in this paper does not necessarily mean distributing all the commodities equally to all the agents. It does mean that unequal allocations should be based on preferences, not on endowments or political advantage.

The study of egalitarianism started in the context of a single commodity. Even though equal division is desirable under unidimensional egalitarianism, it is useful to establish notions to rank the extent of inequality. Sen (1973) establishes the weak equity axiom as an egalitarian notion in the welfare economics context, which requires that an individual who is less able to

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transfer income relative to another should receive more income. Hammond (1976) captures this idea by introducing the equity axiom in the social choice context, and shows that the only class of social orderings that satisfies the equity axiom, Pareto principle, and the symmetry principle (if permutation of incomes makes everyone indifferent, then the two allocations are socially indifferent) is the lexicmin ordering. Hardy et al. (1934) show the equivalence between Lorenz dominance and being dominant by a finite sequence of Pigou-Dalton transfers (Pigou (1912); Dalton (1920)).

A number of current studies, however, have examined a multidimensional context, and researchers generally agree that the multicommodity model cannot be represented by a single dimension. Kolm (1977) first studies the multidimensional dominance issue in the welfare economics context. Marshall and Olkin (1979), however, stress that it is fairly difficult to extend the results of unidimensional models to more dimensions. Fleurbaey and Trannoy (2003) also point out that the standard weak Pareto principle is incompatible with the bundle-reducing transfer principle when agents possibly have different preferences over bundles. In a multicommodity model, Fleurbaey (2005, 2007) and Fleurbaey and Maniquet (2008) study weaker notions of the bundle-reducing transfer principle compatible with the Pareto principle. Sprumont (2012) proposes Consensus, a weaker axiom than the Pareto principle which says that if for any allocations $x$ and $y$ everyone agrees that everyone’s bundle at $y$ is strictly better than that at $x$ then $y$ is strictly preferred to $x$, which is compatible with Dominance Aversion, even a stronger axiom than bundle-reducing transfer principle which states that if $(x_1, \ldots, x_n)$ is a multidimensional allocation, $(y_1, y_2, x_3, \ldots, x_n)$ is socially

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2 The equity axiom (Hammond (1976)) argues that for any allocations $x, y$ and any agents $i, j$, if $j$ benefits more than $i$ in both $x$ and $y$, and if $j$ prefers $y$ to $x$ while $i$ prefers $x$ to $y$ and all the other agents are indifferent between $x$ and $y$, then $y$ should be weakly preferred to $x$ by society.

3 Hammond (1976) refers to this family of social orderings as the lexical difference principle.

4 The bundle-reducing transfer principle says that if $(x_1, \ldots, x_n)$ is a multidimensional allocation, $(x_1, \ldots, x_i - t, \ldots, x_j + t, \ldots, x_n)$ is better than $(x_1, \ldots, x_n)$ whenever $x_j < x_j + t \leq x_i - t < x_i$.

5 This analysis is in line with Sen (1970), who shows in the unidimensional social choice context that Liberalism (for each agent $i$, there is at least one pair of allocations $x, y$ such that $y \succeq_i x$ if and only if $y$ is weakly preferred to $x$ by the society) is incompatible with the Pareto principle.
preferred to \((x_1, \cdots, x_n)\) if \(x_1 < y_1 \leq y_2 < x_2\).

Sprumont (2012) proposes the leximin social ordering as a foundation of consensual egalitarianism. It satisfies both Dominance Aversion and Consensus. However, as Sprumont (2012) points out, this social ordering fails to capture Pareto dominance between certain pairs of allocations. To be specific, it is reasonable to argue that even a social planner who considers the leximin order as a decisive standard may try to Pareto dominate the allocation by permuting bundles,\(^6\) which does not change the leximin ranking of the allocation. The main goal of this paper is to improve on Sprumont (2012) by applying Permutation Pareto principle\(^7\) as an alternative weakened version of the Pareto principle to Consensus, and applying Strong Dominance Aversion which states that if \((x_1, \cdots, x_n)\) is a multidimensional allocation, \((y_1, y_2, x_3, \cdots, x_n)\) is socially weakly preferred to \((x_1, \cdots, x_n)\) if \(x_1 < y_1, y_2 < x_2,\) and \(y_2\) is socially weakly preferred to \(y_1,\)\(^8\) That is, we show that stronger notions of Paretian axioms and the bundle-reducing transfer principle than those used in Sprumont (2012) are applicable.

We propose the leximin Paretian ordering, a social ordering using the leximin ordering as the most important criterion, and considering Permutation Pareto dominance as a secondary standard. In other words, this social ordering endeavors to achieve utilitarianism while obeying the egalitarianism that other social orderings pursue, in the framework of this paper. We justify the leximin Paretian ordering with several axioms.

This paper is organized as follows. Section 2 introduces the model and conditions. Section 3 lays out the axioms and the main results. We provide proofs of propositions and a discussion of the independence of the axioms in the main proposition in an Appendix.

\(^6\)Suppes (1966) first applies permutation criterion in a unidimensional context to introduce the Suppes grading principle, and Saposnik (1981) applies the Suppes principle to the multicommodity context.

\(^7\)The Permutation Pareto Principle states that if permuting bundles can result in every agent preferring her new bundle to the old, then this new allocation should be considered as better.

\(^8\)Strong Dominance Aversion is stronger than Dominance Aversion given that Consensus is satisfied. See section 3 for details.
2. Preliminaries

Let there be a fixed number of commodities $m \geq 2$ in the economy. Let $X = \mathbb{R}^m_+$ be the commodity space. Let $N = \{1, \ldots, n\}$ represent a fixed, finite set of individuals such that $n \geq 2$, and let $X^N$ be the set of conceivable allocations. For any bundle $a \in X$, allocation $x \in X^N$, $1 \leq j \leq m$, and $i \in N$, $a_j$ represents the quantity of the $j$th commodity, $x_i$ represents the $i$th bundle in $x$, and $x_{ij}$ represents the quantity of the $i$th bundle’s $j$th commodity in $x$. For any $a, b \in X$, $a \geq b$ if and only if $a_i \geq b_i$ for all $i$, $a > b$ if and only if $a_i > b_i$ for all $i$ and the inequality is strict for at least one person. Every agent $i \in N$ has a strictly monotonic\footnote{The preference $R_i$ is \textit{monotonic} if for any $a, b \in X$, $b \geq a$ and $b_j > a_j$ for some $j$ implies $bR_ia$.}, continuous\footnote{The preference $R_i$ is \textit{continuous} if for any sequence of pairs $(a^k, b^k)$ where $a^k, b^k \in X$ and $b^kR_i a^k$ for all $k$, and $a^k \to a$ and $b^k \to b$, $bR_ia$.}, and rational\footnote{The preference $R_i$ is \textit{complete} if for all $a, b \in X$ either $aR_ib$, $bR_ia$, or both are true, $R_i$ is \textit{transitive} if for all $a, b, c \in X$ $aR_ib$ and $bR_ic$ implies $aR_ic$, and $R_i$ is \textit{rational} if it is complete and transitive.} preference ordering $R_i$ over $X$. This paper aims to set a social ordering $\mathcal{R}$ over $X^N$. Let $P_i$ denote the strict preference relation associated with $R_i$, and $\mathcal{P}$ denote the strict preference relation associated with $\mathcal{R}$.

We say $\mathcal{R}$ is an ordering over $X$ which agrees with $\cap_{i \in N} R_i$ when for any two bundles $a, b \in X$, $bP_i a$ for all $i \in N$ implies $bPa$. Note that $bR_ia$ for all $i \in N$ also implies $bRa$ for any $a, b \in X$ if $\mathcal{R}$ is continuous since each $R_i$ for any $i \in N$ is strictly monotonic. The term ‘agree with $\cap_{i \in N} R_i$’ was used in Sprumont (2012), which expresses ‘society’s evaluation’ of the relative value of commodity bundles.

For any allocation $x = (x_1, \ldots, x_n) \in X^N$, denote by $(x_1^R, \ldots, x_n^R)$ the allocation obtained by rearranging the bundles $x_1, \ldots, x_n$ from the worst to the best so that agent 1 now has the worst and agent $n$ has the best according to $\mathcal{R}$, that is, $x_n^R \leq x_{n-1}^R \leq \cdots \leq x_1^R$ with a tie-breaking rule as follows: for any $x_i$ and $x_j$ such that $i < j$ and $x_i \geq x_j$, $x_i$ is arranged before $x_j$ in $(x_1^R, \ldots, x_n^R)$. We denote $x^R = (x_1^R, \ldots, x_n^R)$.

For any $x \in X^N$ and any permutation $\pi$ on $N$, we denote $\pi(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)}) \in X^N$. For any $x, y \in X^N$, we say $y$ is permuted from $x$ with a permutation $\pi$ on $N$ if $y = \pi(x)$. We also define Pareto dominance in a formal way: For any $x, y \in X^N$, we say that $y$ Pareto dominates $x$, denoted $y \geq_{\text{par}} x$, when $y_iR_ix_i$ for all $i \in N$, and say that $y$ strictly Pareto
dominates \( x \), denoted \( y \succ_{\text{par}} x \), when \( y_i \succ_{\text{R}_i} x_i \) for all \( i \in N \) and \( y_j \succ_{\text{P}_j} x_j \) for at least one \( j \in N \). We also say that there is a Pareto dominance between \( x \) and \( y \) if \( x \succ_{\text{par}} y \) or \( y \succ_{\text{par}} x \).

3. Lexicographic egalitarianism and the Pareto principle

This section studies classes of social orderings and introduces axioms to provide characterization results.

3.1. Consensual Rawlsian ordering

Sprumont (2012) studies a notion of egalitarianism that comes from Rawls (1971) by introducing a social ordering.

Definition 1. A social ordering \( R \) is a consensual Rawlsian ordering if and only if there is a continuous ordering \( \mathcal{R} \) on \( X \) agreeing with \( \bigcap_{i \in N} \mathcal{R}_i \) such that, for all allocations \( x, y \in X^N \), \( y \succ_x \mathcal{P}_x \) if \( y \succ_1 \mathcal{P}_x \).

We adopt a weak notion of continuity and the Paretian axiom, namely Weak Continuity and Consensus from Sprumont (2012), as the two basic, plausible concepts for society to pursue.

Even though lexicographic social orderings are not continuous (see, for example, Mas-Colell et al. (1995), p. 47 Example 3.C.1.), they do satisfy a weak form of continuity. Weak Continuity requires the social ordering be continuous only for ‘fully egalitarian’ allocations.

Weak Continuity. For any \( a, b \in X \) and any sequence \( \{b_k\} \) in \( X \) converging to \( b \), \( (b^k, \ldots, b^k) \mathcal{R}(a, \ldots, a) \) for all \( k \) implies \( (b, \ldots, b) \mathcal{R}(a, \ldots, a) \).

Weak Continuity is desirable for social orderings given that all the individual preferences are continuous.

Consensus is a Paretian axiom weaker than the standard Pareto principle. Consensus says that an allocation is preferred to another allocation if all the individuals prefer every bundle in the former allocation to that in the latter.

Consensus. For any \( x, y \in X^N \), if \( y_j \succ_{\text{P}_j} x_j \) for all \( i, j \in N \), then \( y \succ_x \).

Dominance Aversion (Sprumont (2012)) is an egalitarian notion which says that reducing bundle dominance is always desirable. That is, reducing the cardinal difference without changing the order of preference is always preferred.
Dominance Aversion. For any $x, y \in X^N$ and any $i, j \in N$, if $x_i > y_i \geq y_j > x_j$ and $y_k = x_k$ for all $k \in N \{i, j\}$, then $yRx$.

Dominance Aversion is justifiable from an egalitarian view point since bundle dominance is recognized as unfair when all the individuals have strictly monotonic preferences, even though it is drastic in that every form of reducing bundle dominance is preferred. Next we introduce an axiom that has much in common with Dominance Aversion. Strong Dominance Aversion says that reducing bundle dominance while retaining the egalitarian order is desirable.

Strong Dominance Aversion. For any $x, y \in X^N$ and any $i, j \in N$, if $x_i > y_i$, $(y_i, \cdots, y_i)R(y_j, \cdots, y_j)$, and $y_j > x_j$ and $y_k = x_k$ for all $k \in N \{i, j\}$, then $yRx$.

Strong Dominance Aversion injects into Dominance Aversion the notion that the society can be more active when it applies the idea of reducing bundle dominance to decide which allocation is more fair. Although Strong Dominance Aversion does not imply Dominance Aversion, Strong Dominance Aversion is ‘stronger’ than Dominance Aversion in the sense that it implies Dominance Aversion under Consensus and Weak Continuity.\textsuperscript{12}

We are now ready to introduce the first proposition.

**Proposition 1.** A social ordering satisfying Consensus, Strong Dominance Aversion, and Weak Continuity is a consensual Rawlsian social ordering.

The proof of Proposition 1 is in Appendix A.1.

Sprumont (2012) introduces Intrinsic Dominance to show that a social ordering satisfying Consensus, Weak Continuity, Dominance Aversion, and Intrinsic Dominance is a consensual Rawlsian social ordering as well as that every simple consensual leximin social ordering satisfies these four axioms. Proposition 1 provides an alternative result by modifying Dominance Aversion to Strong Dominance Aversion and dropping Intrinsic Dominance. It is also straightforward that every simple consensual leximin social ordering satisfies Strong Dominance Aversion.

Intrinsic Dominance is an axiom which argues that if the society values each bundle of an allocation $x$ is at least as much as a bundle $a$, then $x$ must be at least good as $(a, \cdots , a)$.

\textsuperscript{12}For any $a, b \in X$, $a > b$ implies $(a, \cdots , a)P(b, \cdots, b)$ by Consensus, and Weak Continuity enriches the statement to the following: $a \geq b$ implies $(a, \cdots , a)R(b, \cdots , b)$.
**Intrinsic Dominance.** For any \( x \in X^N \) and any \( a \in X \), \((x_i, \cdots, x_i)R(a, \cdots, a)\) for all \( i \in N \) implies \( xR(a, \cdots, a) \).

Intrinsic Dominance contains two main ideas. First, the condition implies that \( \pi(x)R(a, \cdots, a) \) for any permutation \( \pi \) on \( N \) since it only performs a bundle-wise comparison. In other words, the permutation can be ignored in a condition of Intrinsic Dominance. Second, a social ordering can outweigh individual preferences in some special cases. In other words, the social ordering ranks \( x \) better than \( (a, \cdots, a) \) even if \( x \) may be far from an equal division.

We are able to prove that an ordering is a consensual Rawlsian ordering while substituting Strong Dominance Aversion for Dominance Aversion and Intrinsic Dominance, but there is no logical relationship between Strong Dominance Aversion and the other two axioms. Appendix B.1 provides two examples of a social ordering to show that even under Consensus and Weak Continuity, Dominance Aversion and Intrinsic Dominance do not imply Strong Dominance Aversion nor does Strong Dominance Aversion imply Intrinsic Dominance.

### 3.2. Leximin Paretian ordering

Sprumont (2012) introduces an egalitarian social ordering as an extension to the egalitarianism of Rawls (1971).

**Definition 2.** The **leximin extension** to \( X^N \) of an ordering \( R \) on \( X \) is the ordering \( R_{lex}(R) \) on \( X^N \) defined as follows: \((y_1, \cdots, y_n)R_{lex}(R)(x_1, \cdots, x_n)\) if and only if either there exists \( j \in N \) such that \( y_i^Rx_i^R \) for all \( i < j \) and \( y_j^Rx_j^R \), or else \( y_i^Rx_i^R \) for all \( i \in N \). A social ordering \( R \) is a **simple consensual leximin ordering** if and only if it is the leximin extension of some continuous ordering \( R \) agreeing with \( \cap_{i \in N}R_i \), that is, \( R = R_{lex}(R) \).

We consider a social planner whose first priority is egalitarianism, in the form of a leximin egalitarian criterion. Furthermore, given two allocations equally ranked by her leximin egalitarian criterion, the social planner breaks ties according to Pareto efficiency. The following example illustrates this tie-breaking rule:

**Example 1.** Consider \( n = m = 2 \) and let \( X = \mathbb{R}_+^2 \). Let \( u_i : X \to \mathbb{R}_+ \) as follows:

\[
u_1(p, q) = p^{1/3}q^{2/3}
\]
\[ u_2(p, q) = p^{2/3}q^{1/3}. \]

\( u_i \) represents \( R_i \) for \( i = 1, 2 \).

Let \( a, b \in X \) such that \( a = (a_1, a_2) = (8, 1) \) and \( b = (b_1, b_2) = (4, 4) \). Let \( x, y \in X \) such that \( x = (x_1, x_2) = (a, b) = ((8, 1), (4, 4)) \) and \( y = (y_1, y_2) = (b, a) = ((4, 4), (8, 1)) \). Thus, \( u_1(a) = 2 < 4 = u_1(b) \) and \( u_2(a) = u_2(b) = 4 \). Therefore, \( bPa \) and \( aPb \).

Let \( u_k(p, q) = p^kq^{1-k} \) for any \( k \in (0, 1) \), and define the ordering \( R_k \) so that \( u_k \) represents \( R_k \). It is obvious that \( R_k \) is a continuous ordering for any \( k \in (0, 1) \), and we can easily show that \( R_k \) for any \( k \in [1/3, 2/3] \) agrees with \( R_1 \cap R_2 \) as follows. Let any \( c = (c_1, c_2) \in X \) and \( d = (d_1, d_2) \in X \) such that \( cP_d \) and \( cP_d \), and let any \( k \in [1/3, 2/3] \). Then \( u_1(c) > u_1(d), u_2(c) > u_2(d) \). Note that \( 2 - 3k \in [0, 1] \) and \( 3k - 1 \in [0, 1] \). Therefore \( u_1(c)^{2-3k}u_2(c)^{3k-1} > u_1(d)^{2-3k}u_2(d)^{3k-1} \). That is, \( c_1^{1-k}d_1^{1-k} > d_1^{1-k}d_1^{1-k} \), which means \( cP_kd \). Since \( cP_d \) and \( cP_d \) implies \( cP_kd \) for any \( c, d \in X \), \( R_k \) agrees with \( R_1 \cap R_2 \) for any \( k \in [1/3, 2/3] \).

Let \( R = R_{1/2} \) as a continuous ordering agreeing with \( \cap_{i \in N} R_i \). Note that \( bPa \) since \( u(a) = 2^{3/2} < u(b) = 4 \). Then, \( x^R = y^R = (a, b) \). It is trivial that \( x \) and \( y \) are indifferent under the consensual Rawlsian social ordering \( (\text{Sprumont (2012)}) \) and simple consensual leximin ordering \( (\text{Sprumont (2012)}) \) since they are a permutation of each other. However, even though \( x \) and \( y \) are a permutation of each other, \( y \) definitely strictly Pareto dominates \( x \). Therefore, it is natural for even an egalitarian social planner to prefer \( y \) to \( x \).

This motivates the following definition.

**Definition 3.** Given individual preferences \( R_1, \ldots, R_n \), a social ordering \( R \) is a *leximin Pareto ordering* if there exists a continuous ordering \( R \) on \( X \) agreeing with \( \cap_{i \in N} R_i \), such that for all \( x, y \in X \),

(i) \( yP_{lex}(R)x \) \( \Rightarrow \) \( yP_x \),

(ii) \( yI_{lex}(R)x \) and \( y \geq_{par} x \) \( \Rightarrow \) \( yR_x \);

\( yI_{lex}(R)x \) and \( y >_{par} x \) \( \Rightarrow \) \( yP_x \).

We say that \( R \) is a leximin Pareto ordering based on \( R_i \), and denote it as \( R_{\text{lex}}(R) \).

The leximin Pareto ordering has a secondary criterion given by ‘Pareto dominance’ in contrast with a leximin extension of an ordering. It also contains the idea that for any two bundles \( a, b \in X \) that are evaluated equally
For any leximin Paretian ordering $R = R_{lpo}(R)$ with a continuous ordering $R$ agreeing with $\cap_{i \in N} R_i$, $a R b$ should imply $(a, \ldots, a) R (b, \ldots, b)$ so that $R$ can satisfy Weak Continuity. Moreover, since $b P a$ always implies $(b, \ldots, b) P (a, \ldots, a)$, $(a, \ldots, a) R (b, \ldots, b)$ implies $a R b$. That is, $[a R b \iff (a, \ldots, a) R (b, \ldots, b)]$ is a necessary condition for $R_{lpo}(R)$ to satisfy Weak Continuity.

Recall the definition of leximin Paretian extension. Notice that for any $a, b \in X$, $[a R b \iff (a, \ldots, a) R (b, \ldots, b)]$ is guaranteed when $a I b$ implies $(a, \ldots, a) I (b, \ldots, b)$. To make the sentence clearer, $[a P b \Rightarrow (a, \ldots, a) P (b, \ldots, b)]$ and $[(a, \ldots, a) I (b, \ldots, b) \Rightarrow a I b]$ holds by the first part of the definition of leximin Paretian extension, but $[a I b \Rightarrow (a, \ldots, a) I (b, \ldots, b)]$ and $[(a, \ldots, a) P (b, \ldots, b) \Rightarrow a P b]$ are guaranteed only with the second condition of the definition. That is, the second part of the definition of leximin Paretian extension assures that any leximin Paretian ordering satisfies Weak Continuity whenever the binary relation upon which the leximin Paretian ordering is based is continuous.

(Weak) Permutation Pareto Principle (Sprumont (2012)) is a restriction of the standard weak Pareto principle.

(Weak) Permutation Pareto Principle. For any $x \in X^N$ and any permutation $\pi$ on $N$, $x_{\pi(i)} \geq_{\text{par}} x$, for all $i \in N$ implies $\pi(x) \geq_{\text{par}} x$, and $x_{\pi(i)} >_{\text{par}} x$ implies $\pi(x) >_{\text{par}} x$.

We adopt this axiom in a strong way so that the restricted form follows the standard Pareto principle. That is, the (Strong) Permutation Pareto Principle is a modification of the Weak Permutation Pareto Principle. We use the term Permutation Pareto Principle to indicate the Strong Permutation Pareto Principle throughout this paper. It says that if an allocation is permuted to reach Pareto dominance, then the permuted one is preferred to the old one.

Permutation Pareto Principle. For any $x \in X^N$ and any permutation $\pi$ on $N$, $\pi(x) \geq_{\text{par}} x$ implies $\pi(x) R x$, and $\pi(x) >_{\text{par}} x$ implies $\pi(x) P x$.

Example 1 shows the concept of Permutation Pareto Principle. Recall $x = ((8, 1), (4, 4))$, $y = ((4, 4), (8, 1))$ in Example 1. Since $y$ is permuted from $x$ and $y$ Pareto dominates $x$, we can argue that $y$ is strictly preferred to $x$ by the society.
We say $y$ is *consensually permuted* from $x$ with a social ordering $R$ and a permutation $\pi$ on $N$ if $(y_i, \ldots, y_i) \mathcal{I}(x_{\pi(i)}, \ldots, x_{\pi(i)})$ holds for all $i \in N$. Note that consensual permutation includes the regular permutation.

Consensual Permutation Pareto Principle states that even if the ‘social value’ of each bundle is maintained when a new allocation plan is introduced, if permutation to that new allocation can result in Pareto dominance, then the permuted version of the new allocation should be preferred by the society.

**Consensual Permutation Pareto Principle.** For any $x, y \in X^N$ such that $y$ is consensually permuted from $x$ with $R$ and $\pi$, $y \geq_{\text{par}} x$ implies $yR x$, and $y >_{\text{par}} x$ implies $yP x$.

The Consensual Permutation Pareto Principle actively includes society’s preference in the notion of Permutation Pareto Principle. It says that for any two allocations $x, y$ such that there is a permutation so that each pair of bundles $x_{\pi(i)}$ and $y_i$ for all $i \in N$ are ‘socially’ indifferent, if $y$ is (strictly) Pareto dominant to $x$, then $y$ should be (strictly) socially preferred to $x$. Consensual Permutation Pareto Principle is a stronger axiom than Permutation Pareto Principle. Consensual Permutation Pareto Principle adopts the notion of permutation so that it enlarges the pairs of allocations to which the Pareto principle can be applied compared to the Permutation Pareto Principle. More importantly, this axiom is still compatible with Strong Dominance Aversion as well as Dominance Aversion.

The Consensual Permutation Pareto Principle contains the main inspiration for this paper. We first show that given any simple consensual leximin ordering $R_{\text{lex}}(R)$, $y_{\text{lex}}(R)x$ if and only if $y$ is consensually permuted from $x$ with $R_{\text{lex}}(R)$ and a permutation $\pi$ on $N$: i) if $y_{\text{lex}}(R)x$ then $y_iR_i x_i^R$ for all $i \in N$ so that there exists a permutation $\pi$ such that $y_{\pi(i)}$ for all $i \in N$, which implies $(y_i, \ldots, y_i) \mathcal{I}(x_{\pi(i)}, \ldots, x_{\pi(i)})$ for all $i \in N$, ii) if $y$ is consensually permuted from $x$ with $R_{\text{lex}}(R)$ and some $\pi$, then $(y_i, \ldots, y_i) \mathcal{I}_{\text{lex}}(R)(x_{\pi(i)}, \ldots, x_{\pi(i)})$ for all $i \in N$ so that $y_i^R x_i^R$ for all $i \in N$, which implies $y_{\text{lex}}(R)x$. That is, the Consensual Permutation Pareto Principle directly argues that even if two allocations $x, y$ are indifferent according to a simple consensual lex-

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13For any $x, y \in X^N$ and any permutation $\pi$ on $N$, if $y = \pi(x)$ then $(y_i, \ldots, y_i) \mathcal{I}(x_{\pi(i)}, \ldots, x_{\pi(i)})$ for all $i \in N$ and any social ordering $R$. That is, $\pi(x)$ is consensually permuted from $x$ with any social ordering and $\pi$, which means that the Consensual Permutation Pareto Principle implies the Permutation Pareto Principle.
imin ordering, if \( y \) (weakly) Pareto dominates \( x \), then \( y \) should be (weakly) preferred to \( x \).

The Consensual Permutation Pareto Principle is weaker than standard Pareto Dominance even though it implies the Permutation Pareto Principle. We show using the following example why we cannot impose standard Pareto Dominance on a leximin Paretian ordering, as well as a consensual leximin ordering.

**Example 2.** Consider \( n = m = 2 \), \( X = \mathbb{R}^2_+ \) and recall the assumptions on \( R_1, R_2 \), and \( R \) from Example 1. Consider \( R_{lpe}(R) \) be any leximin Paretian ordering based on \( R \). We have \((4,4)P(\sqrt{2}, 8), (4,4)P(8, \sqrt{2}), (\sqrt{2}, 8)P(4, 4), \) and \((8, \sqrt{2})P(4, 4). \) Then, even though \( (\sqrt{2}, 8), (8, \sqrt{2}) \) Pareto dominates \((4,4), (4,4)\), a consensual leximin ordering based on \( R \) instead of individual preferences strictly prefers \((4,4), (4,4)\) to \((\sqrt{2}, 8), (8, \sqrt{2})\), and thus \((4,4), (4,4)\) Pareto dominates \((\sqrt{2}, 8), (8, \sqrt{2})\). This shows that the leximin Paretian ordering’s primary concern is not the Pareto principle.

**Internal Dominance.** For any \( x, x' \in X^N \), \( y_i, y'_i \in X \) and any permutation \( \pi \) on \( N \), if \( xR x', (x_i, \ldots, x_i)I(x'_i, \ldots, x'_i) \) for some \( i \in N \), and \( y_i, y'_i \) for all \( j \in N \), then \( \pi((y_i; x_{-i}))P(y'_i; x'_{-i}). \)

Internal Dominance says that if allocation \( x \) is preferred to \( x' \), and the \( i \)th bundles of \( x \) and \( x' \) are considered as indifferent by society, and if every agent agrees that bundle \( y_i \) is strictly better than bundle \( y'_i \), then \((y_i; x_{-i})\) is strictly preferred to \((y'_i; x'_{-i})\) by society. Moreover, strict preference holds even when we apply any permutation to the modified allocations. This axiom contains a glimpse of an idea about lexicographic preference. This axiom is strong in the sense that it ignores agent \( i \)’s preference when comparing \((y_i; x_{-i})\) and \((y'_i; x'_{-i})\), and it offers a principle related to permutation. It is, however, plausible as a principle that society would follow. The condition requires that society evaluates \( x_i \) and \( x'_i \) equally, \(^{14}\) and not only agent \( i \) but also all the other agents and even society that pursues Consensus agree that \( y_i \) is strictly better than \( y'_i \). Under these conditions, society is convinced that regardless of \( i \)’s preference, \((y_i; x_{-i})\) is definitely strictly better than \((y'_i; x'_{-i})\) so that \( \pi((y_i; x_{-i})) \) is strictly preferred to \((y'_i; x'_{-i})\) with any permutation \( \pi \).

\(^{14}\) \( x_i \) and \( x'_i \) being socially indifferent can be interpreted that agent \( i \) may have strict ranking between \( x_i \) and \( x'_i \) but still \( x_i \) and \( x'_i \) are valued similar.
We are ready to introduce the following proposition which characterizes the leximin Paretian ordering.

**Proposition 2.** A social ordering satisfies Strong Dominance Aversion, Weak Continuity, Internal Dominance, and the Consensual Permutation Pareto Principle if and only if it is a leximin Paretian ordering.

The proof of proposition 2 is in Appendix A.2, and the independence of the four axioms is provided in Appendix B.2. Notice that Consensus does not appear in Proposition 2; Internal Dominance implies Consensus. Also notice that, under Consensus, we apply a stronger axiom than Dominance Aversion, namely Strong Dominance Aversion, as well as an additional axiom, the Consensual Permutation Pareto Principle, which contains a restricted notion of Pareto principle. That is, Proposition 2 contributes to the line of research initiated by Sprumont (2012) through the introduction of a new social ordering, incomparable with his, satisfying a stronger bundle-reducing principle and a stronger Paretian axiom that are mutually compatible.

We finish this section with the following remarks.

Strong Symmetry is a demanding axiom that also plays an essential role in Sprumont (2012), requiring that permuting an allocation in any way cannot make the allocation better or worse according to a social ordering.

**Strong Symmetry.** For any \( x \in X^N \) and any permutation \( \pi \) on \( N \), \( x \text{I}_\pi(x) \).

This axiom indicates that indifference in a leximin ordering should imply indifference in a social ordering. This fact also points out that Strong Symmetry is not compatible with the Permutation Pareto Principle. The difference between Strong Symmetry and the Permutation Pareto Principle as well as the incompatibility of the two axioms are the main difference between the consensual leximin social ordering and the leximin Paretian ordering.

Internal Separability is a separability property that plays a crucial role, along with Strong Symmetry, in Sprumont (2012) in characterizing the simple consensual leximin ordering. In brief, it says that when the society considers \( x_i \) as valuable as \( x_i' \), and \( y_i \) as valuable as \( y_i' \), agent \( i \)'s individual preference can be ignored.

**Internal Separability.** Let \( i \in N \) and let \( x_i, x_i', y_i, y_i' \in X \) be such that \((x_i, \ldots, x_i)\text{I}(x_i', \ldots, x_i'), (y_i, \ldots, y_i)\text{I}(y_i', \ldots, y_i')\). Then, for all \( z_N, z'_N \in X^N \), \((x_i; z_{-i})\text{R}(x_i'; z'_{-i})\) if and only if \((y_i; z_{-i})\text{R}(y_i'; z'_{-i})\).
Sprumont (2012) shows that Consensus, Dominance Aversion, Weak Continuity, Strong Symmetry, and Internal Separability characterize simple consensusal leximin social orderings.

The basic idea of Internal Separability is important in characterizing a lexicographic social ordering. Internal Separability, however, is a strong axiom that is incompatible with the Permutation Pareto Principle under two plausible axioms, Consensus and Weak Continuity. Since the Consensual Permutation Pareto Principle is an essential axiom in this paper, we need to weaken Internal Separability.

We propose Weak Internal Dominance to explain the relation between Internal Dominance and Internal Separability, and how a weakened version of Internal Separability can be compatible with the Permutation Pareto Principle. Weak Internal Dominance stresses that when the society considers $x_j$ as valuable as $x'_j$, there exist $y_j, y'_j$ such that $(y_j; x_{-j})$ and $(y'_j; x'_{-j})$ maintain the ranking of $x$ and $x'$.

**Weak Internal Dominance.** For any $x, x' \in X^N$ such that $x R x'$, $(x_i, \cdots, x_i) I (x'_i, \cdots, x'_i)$ for some $i \in N$, and for any $y'_i \in X$, there exists $y_i \in X$ such that $(y_i; x_{-i}) R (y'_i; x'_{-i})$.

It is easy to show that Internal Separability implies Weak Internal Dominance: it asks only that given an arbitrary $y'_i$, such a $y_i$ exists without demanding any relationship between $y_i$ and $y'_i$, such as $(y_i, \cdots, y_i) I (y'_i, \cdots, y'_i)$. Internal Dominance, one of the crucial axioms in Proposition 2, also implies Weak Internal Dominance. It is stronger than Weak Internal Dominance in a different way from Internal Separability: it specifies the connection between $y_i$ and $y'_i$, and it requires a strict ranking between $\pi(y_i; x_{-i})$ and $(y'_i; x'_{-i})$. 

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Appendix A. Proofs of propositions 1 and 2

Appendix A.1. Proof of proposition 1.

Proof. Let $R$ be a social ordering satisfying Consensus, Strong Dominance Aversion, and Weak Continuity. Note that $R$ also satisfies Dominance Aversion. Define an ordering $R$ on $X$ so that

$$
\text{for any } a, b \in X, bRa \text{ if and only if } (b, \cdots, b) R (a, \cdots, a).
$$

Since $R$ satisfies Weak Continuity, $R$ is also continuous. Moreover, for any two bundles $a, b \in X$ such that $b \succ_i a$ for all $i \in N$, $(b, \cdots, b) R (a, \cdots, a)$ by Consensus, and therefore $bPa$. That is, $R$ is a continuous ordering over $X$ which agrees with $\cap_{i \in N} R_i$.

Let any $x, y \in X^N$ such that

$$
y^R_1 P x_1^R.
$$

We use the following two claims to show $yP x$. We first claim that for any $\varepsilon > 0$

$$
(x_1^R + \varepsilon, \cdots, x_1^R + \varepsilon) P x.
$$

Let any $\varepsilon > 0$ and $\hat{x} \in X$ such that $\hat{x} > x_1 + \varepsilon$ for all $i \in N$. Assume $x_1^R = x_1$ without loss of generality. Let $a_2, a_3, \cdots, a_{n-1}$ such that $\varepsilon / 2 < a_2 < a_3 < \cdots < a_{n-1} < \varepsilon$. Then

$$(x_1 + a_2, x_1 + \varepsilon, \hat{x}, \cdots, \hat{x}) R (x_1 + \varepsilon / 2, \hat{x}, \cdots, \hat{x})$$

by Dominance Aversion. By applying Dominance Aversion repeatedly, $(x_1 + \varepsilon, \cdots, x_1 + \varepsilon) R (x_1 + a_{n-1}, x_1 + \varepsilon, x_1 + \varepsilon, \cdots, x_1 + \varepsilon, \hat{x}) R \cdots R (x_1 + a_2, x_1 + \varepsilon, \hat{x}, \cdots, \hat{x}) R (x_1 + \varepsilon / 2, \hat{x}, \cdots, \hat{x})$. To finish the proof of the claim, $(x_1 + \varepsilon / 2, \hat{x}, \cdots, \hat{x}) P x$ by Consensus.

We also claim that for any $\varepsilon > 0$

$$
y P (y^R_1 - \varepsilon, \cdots, y^R_1 - \varepsilon).
$$

Let any $b_2, b_3, \cdots, b_{n-1}, \varepsilon$ such that $0 < b_2 < b_3 < \cdots < b_{n-1} < \varepsilon / 2$. Additionally, pick any $\hat{y}$ such that $\hat{y} > y_i$ for all $i \in N$. Without loss of generality, assume that $y_1^R = y_1$. Note that $y_i R y_1$ for all $i \in N$, which implies that

\[15\] This claim follows the idea of part of the proof of Proposition 1 in Sprumont (2012).
$(y_i,\cdots,y_i) \mathbf{R} (y_1,\cdots,y_1)$ for all $i \in N$. We have $(y_2,\cdots,y_2) \mathbf{R} (y_1,\cdots,y_1)$ from $y_2 \mathbf{R} y_1$ and by definition of $R$, and also have $\hat{y} > y_2$, $y_1 > y_1 - b_2$. Therefore, by Strong Dominance Aversion, $y \mathbf{R} (y_1 - b_2,\hat{y},y_3,\cdots,y_n)$. By repeated application of Strong Dominance Aversion,

$$y \mathbf{R} (y_1 - b_2,\hat{y},y_3,\cdots,y_n) \mathbf{R} \cdots \mathbf{R} (y_1 - b_{n-1},\hat{y},\cdots,\hat{y},y_n) \mathbf{R} (y_1 - \varepsilon/2,\hat{y},\cdots,\hat{y}).$$

(A.6)

By Consensus, $(y_1 - \varepsilon/2,\hat{y},\cdots,\hat{y}) \mathbf{P} (y_1 - \varepsilon,\cdots,y_1 - \varepsilon)$, hence,

$$y \mathbf{P} (y_1 - \varepsilon,\cdots,y_1 - \varepsilon).$$

(A.7)

Since $R$ is continuous, there exist $\varepsilon_1,\varepsilon_2 > 0$ such that

$$(y_1 - \varepsilon_1) \mathbf{R} (x_1^R + \varepsilon_1).$$

(A.8)

Then we have $(y_1 - \varepsilon_2,\cdots,y_1 - \varepsilon_2) \mathbf{R} (x_1^R + \varepsilon_1,\cdots,x_1^R + \varepsilon_1)$ by (A.1), and also $y \mathbf{P} (y_1 - \varepsilon_2,\cdots,y_1 - \varepsilon_2)$ and $(x_1^R + \varepsilon_1,\cdots,x_1^R + \varepsilon_1) \mathbf{P} x$ by the previous two claims. Therefore $y \mathbf{P} x$, which is the desired result.

$\square$


To start, we first show that Consensus is implied by Internal Dominance.

**Lemma 1.** Internal Dominance implies Consensus.

**Proof.** Take any $x, y \in X^N$ such that $y_i P x_j$ for all $i, j \in N$, and any social ordering $\mathbf{R}$ that satisfies Internal Dominance. Construct $z^1, z^2, \cdots, z^{n-1} \in X^N$ such that $z^1 = (y_1; x_{-1})$, $z^2 = (y_2; z^1_{-2}), \cdots, z^{n-1} = (y_{n-1}; z^{n-2}_{(n-1)})$. Since $y_i P x_j$ for any $i \in N$, $z^1 \mathbf{P} x$ by Internal Dominance. In the same way, we get $y \mathbf{P} z^{n-1} \mathbf{P} z^{n-2} \cdots \mathbf{P} z^1$, and therefore $y \mathbf{P} x$ by transitivity.

$\square$

We show that a rule satisfies Strong Dominance Aversion, Weak Continuity, Internal Dominance, and Consensual Permutation Pareto Principle is a leximin Paretian ordering. Let $\mathbf{R}$ be any social ordering satisfies the four axioms. Note that $\mathbf{R}$ then also satisfies Consensus and Dominance Aversion.

Define an ordering $R$ on $X$ so that for any $a, b \in X$, $bRa$ if and only if $(b, \cdots, b) \mathbf{R} (a, \cdots, a)$. Since $\mathbf{R}$ satisfies Weak Continuity, $R$ is also continuous and therefore $R$ is a continuous ordering. Moreover, for any two bundles
a, b ∈ X such that bPa for all i ∈ N, (b, · · · , b)P(a, · · · , a) by Consensus, and therefore bPa. That is, R is a continuous ordering over X which agrees with ∩i∈NRi. We will show that R is actually a leximin Pareto ordering based on R using Lemmas 2, 3, and 4. Note that Proposition 1 indicates that R is a consensual Rawlsian social ordering based on R.

Lemma 2. For any x, y ∈ XN, if yPlex(R)x then yRPxR.

Proof. Assume that yPlex(R)x. Then there exists j ∈ N such that yR i xR i for all i < j and yRPxR. If j = 1 then yR 1 xR 1, which implies (π2(y))1R(π1(x))1R, and therefore yRPxR since R is a consensual Rawlsian social ordering based on R.

Now, assume j > 1. We claim that yRPxR. Suppose that xR R yR. Let any ε > 0 and any a > yR. Note that (a + ε)P a for all i ∈ N. By Internal Dominance, if xR R yR then (a + ε, xR 2, · · · , xR n)P(a, yR 2, · · · , yR n). It also holds by Internal Dominance that if (a + ε, xR 2, · · · , xR n)P(a, yR 2, · · · , yR n) then (a + ε, a + ε, xR 3, · · · , xR n)P(a, a, yR 3, · · · , yR n). By applying Internal Dominance repeatedly, we have (a + ε, · · · , a + ε, xR j, · · · , xR n)P(a, · · · , a, yR j, · · · , yR n). Let x = (a + ε, · · · , a + ε, xR j, · · · , xR n), y = (a, · · · , a, yR j, · · · , yR n). On the other hand, since aPyR i for all i ∈ N, xR 1 = xR 1 and yR 1 = yR 1 holds, and since yRPxR and R is a consensual Rawlsian social ordering, yP x also holds, which is a contradiction to the result from repeated application of Internal Dominance.

We complete the first part of the proof by showing that R is a leximin Pareto ordering based on R. Take any leximin Pareto ordering Rlpo(R) that satisfies the following: for any x, y ∈ X such that yPlex(R)x and there is no Pareto dominance between x and y, yR if and only if yRlpo(R)x. We show that R = Rlpo(R) with the following Lemmas 3 and 4.

Lemma 3. yPlex(R)x implies yPx.

Proof. From yPlex(R)x, there exists ℓ ∈ N such that yR i xR i for all i < ℓ and yR ℓ xR ℓ. Suppose that yj = yR and xj = xR.

Fix any ε1 > 0, ε2 > 0 and a = yR − (ε1, · · · , ε1), b = xR + (ε2, · · · , ε2) such that yRPbPb xR. Note that we can find such a and b since R is a continuous ordering. Define z, w ∈ XN as zi = yi for all i ̸= j and zj = a, wj = xj for all i ̸= k and wk = b.

Pick π1, π2 ∈ Π such that π1(z) = zR and π2(w) = wR. Note that yjPizj for all i ∈ N. Therefore, by Internal Dominance, (yj; y−j)Pπ1(zj; y−j), that
is, \( yPz^R \). Moreover, since \( w_kP_x, \pi_2(w_k; x_{-k})P(x_k; x_{-k}) \), that is, \( w^R P x \). We are done if we show \( z^R P w^R \).

Note that \( zR_i \) implies \( (x_j, \ldots, x_j)Ilex(R)(x'_j, \ldots, x'_j) \), which in turn implies \( x_jIx'_j \). Then, from Lemma 6, \( zP_{lex}(R)w \), which implies, by Lemma 2, \( z^R P w^R \).

\[ \square \]

**Lemma 4.** \( yI_{lex}(R)x \) and \( y \geq_{par} x \) implies \( yRx \), \( yI_{lex}(R)x \) and \( y >_{par} x \) implies \( yPx \).

**Proof.** Let any \( x, y \in X^N \) such that \( yI_{lex}(R)x \). Then, there exists a permutation \( \pi \) on \( N \) such that \( y_iIx_{\pi(i)}(i) \) for all \( i \in N \). By the definition of \( R \), \( (y_i, \ldots, y_i)I(x_{\pi(i)}, \ldots, x_{\pi(i)}) \) for all \( i \in N \). That is, \( y \) is consensually permuted from \( x \) with \( R \) and \( \pi \). Then, \( yRx \) holds if \( y \geq_{par} x \) and \( yPx \) holds if \( y >_{par} x \) by the Consensual Permutation Pareto Principle.

\[ \square \]

We finish the proof with Lemma 5, 6, and 7 by showing that any leximin Paretoian ordering satisfies the four axioms in Proposition 2. Let \( R \) be any continuous ordering on \( X \) agreeing with \( \cap_{i \in N} R_i \), and let \( R_{lpo}(R) \) be any leximin Paretoian ordering based on \( R \).

**Lemma 5.** \( R_{lpo}(R) \) satisfies Strong Dominance Aversion.

**Proof.** Let any \( x, y \in X^N \) such that \( x_i > y_i \), \( (y_i, \ldots, y_i)R_{lpo}(R)(y_j, \ldots, y_j) \), and \( y_j > x_j \) for some \( i, j \in N \) and \( y_k = x_k \) for any \( k \in N \{i, j\} \). For any \( a, b \in X \), \( bPa \) implies \( (b, \ldots, b)P_{lex}(R)(a, \ldots, a) \); hence, \( (b, \ldots, b)P_{lpo}(R)(a, \ldots, a) \). Therefore, \( (a, \ldots, a)R_{lpo}(R)(b, \ldots, b) \) implies \( aRb \); hence, \( y_iR_{lj} \). Since all \( R_k \)’s for any \( k \in N \) are strictly monotonic, \( x_i y_i R_{lj} P x_j \) holds. From the fact that \( R_{lex}(R) \) is a lexicographic extension of \( R \), and since \( y_iIx_k \) for all \( k \in N \{i, j\} \) along with \( x_i y_i R_{lj} P x_j \), \( yP_{lex}(R)x \) holds, which implies \( yR_{lpo}(R)x \).

\[ \square \]

**Lemma 6.** \( R_{lpo}(R) \) satisfies Internal Dominance.

**Proof.** Take any \( x, x' \in X^N \) such that \( (x_j, \ldots, x_j)I_{lpo}(R)(x'_j, \ldots, x'_j) \) for some \( j \in N \), any \( y_j, y'_j \in X \) such that \( y_jP_{lj} y'_j \) for all \( i \in N \), and any permutation \( \pi \) in \( N \). Further assume that \( xR_{lpo}(R)x' \). Then \( xR_{lex}(R)x' \) also holds. Because \( y_jP_{lj} y'_j \) for all \( i \in N \), \( y_jP_{lj} y'_j \). Also, \( (x_j, \ldots, x_j)I_{lpo}(R)(x'_j, \ldots, x'_j) \) implies \( (x_j, \ldots, x_j)I_{lex}(R)(x'_j, \ldots, x'_j) \), which in turn implies \( x_jIx'_j \). Then, from...
the fact that \( R_{lex}(R) \) is the leximin extension of \( R \) along with \( x R_{lex}(R) x' \), \( x_j I' \) and \( y_j P' \), \( (y_j; x_{-j}) P_{lex}(R)(y'_j; x'_{-j}) \) holds. Note that \( (\pi((y_j; x_{-j}))) R = (y_j; x_{-j}) R \). Then it is immediate that \( \pi((y_j; x_{-j})) P_{lex}(R)(y'_j; x'_{-j}) \), which in turn implies that \( \pi((y_j; x_{-j})) P_{lpo}(R)(y'_j; x'_{-j}) \).

Lemma 7. \( R_{lpo}(R) \) satisfies the Consensual Permutation Pareto Principle.

Proof. Let any \( x, y \in X^N \) such that \( y \) is consensually permuted from \( x \) with \( R_{lpo}(R) \) and \( \pi \). We need to show that \( y \geq_{par} x \) implies \( y R_{lpo}(R) x \) and \( y >_{par} x \) implies \( y P_{lpo}(R)x \). That is, if we show \( y I_{lex}(R)x \) then we are done. We have \( (y_i \cdots, y_i) I_{lpo}(R)(x_{\pi(i)} \cdots, x_{\pi(i)}) \) for all \( i \in N \), which implies \( (y_i \cdots, y_i) I_{lex}(R)(x_{\pi(i)} \cdots, x_{\pi(i)}) \), and therefore \( y_i x_{\pi(i)} \) for all \( i \in N \). Thus, \( y I_{lex}(R) \pi(x) \) holds, and \( y I_{lex}(R)x \) also holds since \( (\pi(x)) R = x R \).

From the fact that Weak Continuity holds immediately from the continuity of \( R \), and by Lemmas 5, 6, and 7, the leximin Pareto ordering satisfies the four axioms.

Appendix B. Independence of the axioms.

Appendix B.1. Strong Dominance Aversion, Dominance Aversion, and Intrinsic Dominance

We show that even under Consensus and Weak continuity, Strong Dominance Aversion and the combination of Dominance Aversion and Intrinsic Dominance do not imply each other by providing two examples of social orderings in the following two paragraphs.

We first construct an example of a social ordering that satisfies Consensus, Weak Continuity, Dominance Aversion, and Intrinsic Dominance, but fails to satisfy Strong Dominance Aversion. Let \( R, R' \) be continuous orderings on \( X \) such that \( R \) agrees with \( \cap_{i \in N} R_i \), and \( R' \) satisfies the following: for any \( a, b \in X \), \( b > a \) implies \( b P' a \).\(^{16}\) Construct a social ordering \( R' \) as follows: for any \( x, y \in X^N \) \( y R' x \) if and only if either i) \( y_i R x_i \), ii) \( y_i R x_i \) and \( x \) is an equal division, or iii) \( y_i R x_i \) and \( x \) is not an equal division, and \( y R_{lex}(R')x \). Then \( R' \) clearly satisfies Consensus from part i) of the definition of \( R \), and

\(^{16}\)Since \( R' \) is continuous, it follows that \( b \geq a \) implies \( b R' a \).
We show by the following numerical example that \( R' \) satisfies Dominance Aversion, but fails to satisfy Strong Dominance Aversion. Consider \( n = 3, m = 2, u_i(a) = a_1 + a_2 \) represents \( R \), for all \( i = 1, 2, 3 \). \( R = R_1 \), and \( u(a) = 10a_1 + a_2 \) represents \( R' \). We first show that \( R' \) satisfies Dominance Aversion. Let any \( x, y \in X^N \) such that \( x_i > y_i \geq y_j > x_j \) and \( x_k = y_k \). We are done if we show \( y \mathrel{R'} x \). Note that \( x \) is not an equal division from the fact that \( x_i > x_j \). If \( x_j < x_k \) then \( y^R_i P^R x^R_1 \), which implies \( y^R x \). If \( x_j \geq x_k \) then \( y^R_i I^R x^R_1 \). Moreover, \( x_i > y_i \geq y_j > x_j \geq x_k = y_k \) implies \( x_i P^R y_i R^R y_j P^R x_j R^R x_k P^R y_k \), thus \( y \mathrel{R'} x \). To show that \( R' \) violates Strong Dominance Aversion, let two allocations \( x = ((2, 2), (3, 3), (10, 10)) \) and \( y = ((2, 2), (4, 4), (1, 9)) \). Note that neither \( x \) nor \( y \) is an equal division, and \( x^R_1 = y^R_1 = (2, 2) \), and \( x^R_2 = (2, 2) P^R(1, 9) = y^R_2 \). Also notice that \( x_3 > y_3, (y_3, \ldots, y_3) R^R(y_2, \ldots, y_2) \) from \( y_3 P^R y_2, y_2 > x_2, \) and \( y_1 = x_1 = (2, 2) \). However, \( x P^R y \) since \( x^R_1 y^R_1 \) and \( x P_{lex}(R') y \), which violates Strong Dominance Aversion.

A leximin Paretian ordering \( R_{lpo}(R) \) with a continuous ordering \( R \) agreeing with \( \cap_{i \in N} R_i \) is an example that satisfies Consensus, Weak Continuity, and Strong Dominance Aversion, but fails to satisfy Intrinsic Dominance. Proposition 2 shows that \( R_{lpo}(R) \) satisfies the first three axioms. We show that \( R_{lpo}(R) \) violates Intrinsic Dominance using the following example. Pick any \( a, b \in X \) such that \( b, \ldots, b \mathrel{I_{lpo}(R)}(a, \ldots, a) \) but \( b P_{lpo} a \). Then \( b I a \) also holds. Intrinsic Dominance argues that \( (b, a, \ldots, a) I_{lpo}(R)(a, \ldots, a) \) must hold. Since we have \( (b, a, \ldots, a) I_{lex}(R)(a, \ldots, a) \) from \( a P b \) and \( (b, a, \ldots, a) \succ_{par} (a, \ldots, a) \) from \( b P_{lpo} a \), however, \( (b, a, \ldots, a) P_{lpo}(R)(a, \ldots, a) \), which violates Intrinsic Dominance.

Appendix B.2. Axioms in proposition 2

The following four examples show that the axioms in Proposition 2 are independent.

Let \( R \) be a continuous ordering over \( X \) agreeing with \( \cap_{i \in N} R_i \), and \( R_{lpo}(R) \) be any leximin Paretian ordering based on \( R \).

1) Define \( y \mathrel{R_{lax}(R)} x \) if and only if either there exists \( j \in N \) such that \( y_j P x_j \) and \( y^R_i I^R x^R_i \) for all \( i > j \), or \( y^R_i I^R x^R_i \) for all \( i \in N \). That is, \( R_{lax}(R) \) is a ‘leximax’ extension to \( X^N \) of an ordering \( R \) on \( X \). Now, define \( R_{DA} \) as follows: \( y \mathrel{R_{DA}} x \) if and only if either \( y \mathrel{R_{lax}(R)} x \), or \( y \mathrel{I_{lax}(R)} x \) and \( y \geq_{par} x \). Then \( R_{DA} \) is an example that violates Strong Dominance Aversion only.
Let any \( x, y \in X^N \) such that there are \( i, j \in N \) such that \( x_i > y_i \geq y_j > x_j \) and \( y_k = x_k \) for all \( k \in N \setminus \{i, j\} \). Then \( xP_{DA}y \) since \( xP_{tax}(R)y \), which implies that \( P_{DA} \) violates Dominance Aversion, and therefore also violates Strong Dominance Aversion. It is trivial that all the other axioms are satisfied.

2) Define an ordering \( \bar{R} \) on \( X \) as follows: \( y \bar{R} x \) if and only if either \( y \bar{P}_1 x \), or \( y \bar{I}_1 x \) and \( y_1 \geq x_1 \). Then \( \bar{R} \) is a discontinuous ordering agreeing with \( \cap_{i \in N} R_i \). Now define \( R_{WC} \) as any leximin Paretian ordering based on \( \bar{R} \). Then, \( R_{WC} \) violates Weak Continuity only.

3) If \( n \geq 3 \), define \( R_{ID} = R_{lex}(R) \). If \( n = 2 \), define \( R_{ID} \) as follows: \( yR_{ID}x \) if either \([y_1^R P x_1^R]\), \([y_1^R I x_1^R] \) and \( y \geq_{par} x \), or \([y_1^R I x_1^R] \) and \( y^R =_{par} x^R \).

All the axioms are satisfied when \( n \geq 3 \). Assume \( n = 2 \) for the rest of this example, and let \( x_1, y_1, a, b \in X \) such that \( bP_1 a, aP_2 b \), and \( y_1 P bP aP x_1 \). Then \((x_1, a)P_{ID}(x_1, b)\) since \((x_1, a)^R = (x_1, a), (x_1, b)^R = (x_1, b)\), and \( aP_2 b \). Moreover, \((y_1, b)P_{ID}(y_1, a)\) for any \( y_1 \in X \) since \((y_1, b)^R = (b, y_1)\) and \( bPa \). That is, even though \((x_1, a)P_{ID}(x_1, b)\) holds, there is no bundle \( y_1' \in X \) that satisfies \((y_1', a)R_{ID}(y_1, b)\), which indicates that \( R_{ID} \) does not satisfy Internal Dominance.

We need to show that \( R_{ID} \) satisfies all the other axioms. It is trivial that \( R_{ID} \) satisfies Strong Dominance Aversion and Weak Continuity. To show is that \( R_{ID} \) satisfies Consensual Permutation Pareto Principle, let any \( x, y \in X^N \) such that \( y \) is consensually permuted from \( x \) with \( R_{ID} \) and \( \pi \). Then we also have \( yI_{\pi(i)} x \) for \( i = 1, 2 \), and thus \( x_1^\pi, y_1^\pi \) holds. That is, \( yP_{ID}x \) if \( y >_{par} x \) and \( yR_{ID}x \) if \( y \geq_{par} x \). Therefore \( R_{ID} \) satisfies the Consensual Permutation Pareto Principle.

4) An example that violates the Consensual Permutation Pareto Principle only is \( R_{lex}(R) \). Sprumont (2012) shows that \( R_{lex}(R) \) satisfies Consensus, Dominance Aversion, and Weak Continuity. Then \( R_{lex}(R) \) satisfies Strong Dominance Aversion is trivial. However, the Permutation Pareto Principle is violated since for any \( x \in X^N \) and any permutation \( \pi \) on \( N \) such that \( \pi(x) >_{par} x \), \( \pi(x)I_{lex}(R)x \).

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